Stability Analysis of an *SVIR* Epidemic Model with Nonlinear Incidence Rate

การวิเคราะห์เสถียรภาพของแบบจำลองการระบาด *SVIR* ที่มีอัตราอุบัติการณ์ที่ไม่เป็นเชิงเส้น

Aphirak Boonpikum (อภิรักษ์ บุญพิคำ)* Dr.Wirawan Chinviritasit (ดร.วิราวรรณ ชินวิริยสิทธิ์)**

ABSTRACT

This paper, an *SVIR* epidemic model with nonlinear incidence rate is investigated. The stability of the model depends on the basic reproduction number. The global asymptotically stable of the disease-free is proved by constructing a Lyapunov function and LaSalle's invariant set theorem. An unique endemic equilibrium exists and is locally asymptotically stable whenever $R_0 > 1$. Numerical simulation is illustrated to support the analytical results and to study the effect of nonlinear incidence rate on the *SVIR* model.

บทคัดย่อ

งานวิจัยนี้ได้สำรวจแบบจำลองการระบาด *SVIR* ที่มีอัตราอุบัติการณ์ที่ไม่เป็นเชิงเส้น ซึ่งเสถียรภาพของแบบจำลองนี้ขึ้นอยู่กับค่าระดับการติดเชื้อของโรค การสร้างฟังก์ชันไลอะพอนอฟและทฤษฎีเซตไม่แปรผันของลาซาลได้นำไปพิสูจน์เสถียรภาพเชิงเส้นกำกับวงกว้างของจุดสมดุลภายใต้สภาวะไร้โรค และจุดสมดุลภายใต้สภาวะการแพร่ระบาดมีเสถียรภาพเฉพาะที่เชิงเส้นกำกับเมื่อ $R_0 > 1$ นอกจากนี้งานวิจัยได้แสดงผลลัพธ์เชิงตัวเลขเพื่อใช้สนับสนุนผลลัพธ์เชิงทฤษฎีและศึกษาผลกระทบของอัตราอุบัติการณ์ไม่เชิงเส้นในแบบจำลอง *SVIR*

Key Words: *SVIR* , Vaccination, Lyapunov function

คำสำคัญ: แบบจำลองการระบาด *SVIR* การให้วัคซีนฟังก์ชันไลอะพอนอฟ

---

* *Student, Master of Science, Program in Applied Mathematics, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi*

** *Associate Professor, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi*

1. Introduction

Mathematical models in epidemiology or epidemic model are used to understand epidemiological phenomena, to predict the infectious disease progress and to prevent public health for controlling the diseases. Most epidemic models descend from the classical SIR (Susceptible-infectious-recovered) model of Kermack and Mckendric established in 1927 (W.O. Kermarck and A.G. Mckendrick, 1927). In fact, vaccines are extremely important and have been proved to be the most effective and cost-efficient method of preventing infectious diseases such as measles, polio, diphtheria, tetanus, pertussis, and tuberculosis. In recent years, many researchers have discussed the SIR model allowing vaccination, that is, the $SVIR$ model.

Kribs-Zaleta C.M. and Velasco-Hernández J.X. (2000) presented a $SVIS$ epidemic model and found that this model exhibited the phenomenon of backward bifurcation. Furthermore, the model is applied for studying the vaccination of disease pertussis and tuberculosis. Liu X., Takeuchi Y. and Iwami S. (2008) established two $SVIR$ model to describe continuous vaccination strategy and pulse vaccination strategy, respectively. However, in modeling disease epidemics take place in ecological system, the incidence rate plays a key factor in the transmission of infectious diseases. It has been suggested by several authors that the disease transmission process may have a nonlinear incidence rate. This allows one to include behavioral changes and prevent unbounded contact rates. A particular example of such an incidence rate is given by $\beta I(1+\nu I^{k-1})S$, where $\beta > 0$, $\nu > 0$ and $k > 0$. This incidence with ν near one, represents saturation or multiple exposures before infection. van der Driessche P. and Watmough J. (2000, 2003) introduced this incident into SIS epidemic model where $\beta > 0$, $\nu > 0$ and $k > 0$. They found that SIS epidemic model with this incidence rate exhibited backward bifurcation. Alexander M.E. and Moghadas S.M. (2004) analyzed an SIV model with a generalized nonlinear incidence rate $\beta I(1+\nu I^{q})S$, where $\beta > 0$, $\nu \geq 0$ and $0 < q \leq 1$. The results showed that bistability and various Hopfbifurcation occurred. Jin Y., Wang W. and Xiao S. (2007) studied the backward bifurcation, the Hopf bifurcation and Bogdanov-Takens bifurcation by analyzing a $SIRS$ model with nonlinear incident rate in the form $\beta I(1+\nu I^{k-1})S$, where $\beta > 0$, $\nu > 0$ and $k = 2$. Zhou X. and Cui J. (2011) investigated the $SEIV$ model with nonlinear incidence rate of the form $\beta I(1+\nu I^{q-1})S$ with $\beta > 0, \nu > 0$ and $q = 2$.

The aim of this paper is to investigate the effect of nonlinear incidence rate $\beta SI(1+\nu I)$, $\beta > 0$ and $0 < \nu \leq 1$ on the dynamic of a $SVIR$ epidemic model proposed by Liu X., Takeuchi Y. and Iwami S. (2008). To this end, we consider the following differential equations

$$\begin{aligned}
\frac{dS}{dt} &= \mu - \mu S - \beta SI(1+\nu I) - \alpha S, \\
\frac{dV}{dt} &= \alpha S - \beta_1 VI - \gamma_1 V - \mu V, \\
\frac{dI}{dt} &= \beta SI(1+\nu I) + \beta_1 VI - \gamma I - \mu I, \\
\frac{dR}{dt} &= \gamma_1 I + \gamma I - \mu R,
\end{aligned} \tag{1}$$

where the state variables S, V, I and R denote the densities (or fractions) of susceptible, vaccinated, infected and recovered individuals, respectively. Theparameters μ is the natural birth rate which is assumed to be equal natural death rate; β is the transmission rate; α is the rate at which susceptible individuals who are moved into the vaccination process; β_1 is the possibility of infection with a disease transmission rate while contracting with

infected individuals; γ is the recovery rate of infected individuals and γ_1 is the average rate for individuals to obtain immunity andmove into recovered population, respectively. It is assumed that β_1 is less than β because the vaccinated individuals may have some partial immunity during the process or they may recognize the transmission characters of the disease and hence decrease the effective contacts with infected individuals.Adding all equations in (1) gives

$$\frac{dN}{dt}=\frac{dS}{dt}+\frac{dV}{dt}+\frac{dI}{dt}+\frac{dR}{dt}=\mu-\mu N \qquad (2)$$

which has the following implication: the three-dimension simplex

$$\Gamma=\left\{(S,V,I,R)\in\Re_+^4 : S+V+I+R=1\right\}$$

is positively invariant. On the simplex Γ, we have $R(t)=1-S(t)-V(t)-I(t)$, thus, the dynamics of system (1) is studied by analyzing the following three-dimensional system:

$$\begin{aligned}\frac{dS}{dt}&=\mu-\mu S-\beta SI(1+\nu I)-\alpha S,\\ \frac{dV}{dt}&=\alpha S-\beta_1 VI-\gamma_1 V-\mu V,\\ \frac{dI}{dt}&=\beta SI(1+\nu I)+\beta_1 VI-\gamma I-\mu I,\end{aligned} \qquad (3)$$

The dynamical behavior of (1) on Γ is equivalent to that of (3). Therefore, the system (3) is studied in the feasible region

$$\begin{aligned}\Gamma=\Big\{&S,V,I)\in\square_+^3 : 0\le S,V,I\le 1,\\ &0\le S+V+I\le 1\Big\}\end{aligned} \qquad (4)$$

where $\square_+^3$ denotes the non-negative cone of $\square^3$ including its lower-dimensional faces.

The paper is organized as follows: In section 2, the model is analyzed the existence of equilibriums in the model. The SVIR model is conserved to analyze the stability analysis which consists of the disease-free equilibrium and endemic equilibria. The Lyapunovfunction is analyzed for proving the global stabilityunder the condition and the locally asymptotically stable is analyzed both equilibriums in Section 3. The last section, Numerical simulations are performed in the last section.

2. Existence of Equilibriums

In the absence of infection, by setting the right-hand side of system (3) to zero and solving the obtained results, the model has a disease-free equilibrium E_0,

$$\begin{aligned}E_0&=(S_0,V_0,I_0)\\ &=\left(\frac{\mu}{\mu+\alpha},\frac{\alpha\mu}{(\mu+\alpha)(\mu+\gamma_1)},0\right)\end{aligned} \qquad (5)$$

and an endemic equilibrium E^*, with

$$\begin{aligned}S^*&=\frac{\mu}{\mu+\alpha+\beta I^*(1+\nu I^*)},\\ V^*&=\frac{\alpha S^*}{\mu+\gamma_1+\beta_1 I^*},\end{aligned} \qquad (6)$$

where I^* is positive root of cubic equation

$$h(I^*)=wI^{*3}+xI^{*2}+yI^*+z=0, \qquad (7)$$

with

$$\begin{aligned}w&=\nu\beta\beta_1(\mu+\gamma)>0,\\ x&=\nu\beta(\mu+\gamma)(\mu+\gamma_1)+\gamma\beta\beta_1\\ &\quad+(1-\nu)\mu\beta\beta_1>0,\\ y&=(\mu+\gamma)(\beta(\mu+\gamma_1)+\beta_1(\mu+\alpha))\\ &\quad-\nu\beta\mu(\mu+\gamma_1)-\mu\beta\beta_1,\\ z&=(\mu+\gamma_1)(\mu+\alpha)(\mu+\gamma)(1-R_0),\\ R_0&=\frac{\beta\mu}{(\mu+\alpha)(\mu+\gamma)}+\frac{\beta_1\alpha\mu}{(\mu+\alpha)(\mu+\gamma)(\mu+\gamma_1)}.\end{aligned} \qquad (8)$$

Here, R_0 is called the basic reproduction number of system (3). Thus, the following Theorem is established.

Theorem 1. The system (3) has a unique endemic equilibrium E^* for $R_0 > 1$, and has no endemic equilibrium for $R_0 \le 1$.

Proof. Since $0 < v \le 1$. It is clear that, from (8) $w, x > 0$ and Then, the roots of (7) depend on the sign of y and z .Now, two cases of R_0 are considered as follows:

Case 1: $R_0 \le 1$, then

$$\begin{aligned} y &= (\mu+\gamma)(\beta(\mu+\gamma_1)+\beta_1(\mu+\alpha)) - v\beta\mu(\mu+\gamma_1) \\ &\quad - v\beta\mu(\mu+\gamma_1) - \mu\beta\beta_1 \\ &> \beta\mu(\mu+\gamma_1)+\beta_1(\mu+\gamma)(\mu+\alpha) - \beta\mu(\mu+\gamma_1) \\ &\quad - \beta\mu(\mu+\gamma_1) - \mu\beta\beta_1 \\ &> \beta_1(\mu+\gamma)(\mu+\alpha)\left(1-\frac{\mu\beta}{(\mu+\gamma)(\mu+\alpha)}\right. \\ &\quad \left.-\frac{\alpha\mu\beta_1}{(\mu+\gamma_1)(\mu+\alpha)(\mu+\gamma)}\right) \\ &= \beta_1(\mu+\gamma)(\mu+\alpha)(1-R_0) > 0, \\ z &= (\mu+\gamma_1)(\mu+\alpha)(\mu+\gamma)(1-R_0) > 0. \end{aligned}$$

Thus, if $R_0 \le 1$, we have $w > 0$, $x > 0$, $y > 0$ and $z > 0$. By the Descartes's rules of signs, there is no endemic equilibrium.

Case 2: $R_0 > 1$, then

$$\begin{aligned} y &= \beta(\mu+\gamma)(\mu+\gamma_1)+\beta_1(\mu+\gamma)(\mu+\alpha) \\ &\quad - v\beta\mu(\mu+\gamma_1) - \mu\beta\beta_1 \\ &= \beta(\mu+\gamma)(\mu+\gamma_1)\left(1-\frac{v\mu}{(\mu+\gamma)}\right) \\ &\quad + \beta_1(\mu+\gamma)(\mu+\alpha)\left(1-\frac{\mu\beta}{(\mu+\gamma)(\mu+\alpha)}\right) \\ &> \beta\mu(\mu+\gamma_1)\left(1-\frac{v\mu}{(\mu+\gamma)}\right) \\ &\quad + \beta_1\mu\alpha\left(1-\frac{\mu\beta}{(\mu+\gamma)(\mu+\alpha)}\right) \\ &> (\mu+\gamma)(\mu+\alpha)(\mu+\gamma_1)\left(\frac{\beta\mu}{(\mu+\gamma)(\mu+\alpha)}\right. \\ &\quad \times\left(1-\frac{v\mu}{(\mu+\gamma)}\right)+\frac{\beta_1\mu\alpha}{(\mu+\gamma)(\mu+\alpha)(\mu+\gamma_1)} \\ &\quad \left.\times\left(1-\frac{\mu\beta}{(\mu+\gamma)(\mu+\alpha)}\right)\right) \end{aligned}$$

Since

$$R_0 = \frac{\mu\beta}{(\mu+\gamma)(\mu+\alpha)}+\frac{\beta_1\mu\alpha}{(\mu+\gamma)(\mu+\alpha)(\mu+\gamma_1)} > 1,$$

we get

$$\frac{\beta_1\mu\alpha}{(\mu+\gamma)(\mu+\alpha)(\mu+\gamma_1)} > 1-\frac{\mu\beta}{(\mu+\gamma)(\mu+\alpha)}.$$

It follows that

$$\begin{aligned} y &> (\mu+\gamma)(\mu+\alpha)(\mu+\gamma_1)\left(\frac{\beta\mu}{(\mu+\gamma)(\mu+\alpha)}\left(1-\frac{v\mu}{(\mu+\gamma)}\right)\right. \\ &\quad \left.+\left(1-\frac{\mu\beta}{(\mu+\gamma)(\mu+\alpha)}\right)\left(1-\frac{\mu\beta}{(\mu+\gamma)(\mu+\alpha)}\right)\right) \\ &= (\mu+\gamma)(\mu+\alpha)(\mu+\gamma_1)\left(\frac{\beta\mu}{(\mu+\gamma)(\mu+\alpha)}\left(1-\frac{v\mu}{(\mu+\gamma)}\right)\right. \\ &\quad \left.+\left(1-\frac{\mu\beta}{(\mu+\gamma)(\mu+\alpha)}\right)^2\right) \end{aligned}$$

and $z = (\mu+\gamma_1)(\mu+\alpha)(\mu+\gamma)(1-R_0) < 0$.

Thus, $w > 0$, $x > 0$, $y > 0$ and $z < 0$ if $R_0 > 1$. By the Descartes's rules of signs, there is an unique positive endemic equilibrium.This completes the proof.

3. Stability of equilibrium

Theorem 2. The disease-free equilibrium E_0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Proof. The Jacobian of system (3) evaluated at E_0 is

$$J(E_0) = \begin{bmatrix} -(\mu+\alpha) & 0 & \dfrac{-\beta\mu}{\mu+\alpha} \\ \alpha & -(\mu+\gamma_1) & \dfrac{-\beta_1\mu\alpha}{(\mu+\alpha)(\mu+\gamma_1)} \\ 0 & 0 & (\mu+\gamma)(R_0-1) \end{bmatrix} \quad (9)$$

with eigenvalues

$$\lambda_1 = -(\mu+\alpha) < 0,\ \lambda_2 = -(\mu+\gamma_1) < 0 \quad \text{and } \lambda_3 = (\mu+\gamma)(R_0-1). \tag{10}$$

It follows that all eigenvalues (10) are negative real part. Therefore, the disease-free equilibrium E_0 is locally asymptotically stable.

Theorem 3. If $R_{01} \le 1$, then the disease-free equilibrium E_0 of system (3) is globally asymptotically stable, where

$$R_{01} = \frac{(\beta\mu(\mu+\gamma_1)+\beta_1\alpha\mu)(1+\nu)}{(\mu+\alpha)(\mu+\gamma)(\mu+\gamma_1)}.$$

Proof. Consider a Lyapunov function

$$L(S,V,I) = S - S_0 - S_0\ln\left(\frac{S}{S_0}\right) + V - V_0 - V_0\ln\left(\frac{V}{V_0}\right) + I,$$

its derivative along the solution of the system (3) is

$$\begin{aligned}
\frac{dL}{dt} &= \frac{dS}{dt} + \frac{dV}{dt} + \frac{dI}{dt} - \frac{S_0}{S}\cdot\frac{dS}{dt} - \frac{V_0}{V}\cdot\frac{dV}{dt} \\
&= \mu - \mu S - \mu V - \mu I - \gamma_1 V - \gamma I - \frac{\mu S_0}{S} \\
&\quad + \mu S_0 + \beta S_0 I(1+\nu I) + \alpha S_0 - \frac{\alpha S V_0}{V} \\
&\quad + \beta_1 V_0 I + \gamma_1 V_0 + \mu V_0 \\
&= -\mu S - \frac{\mu\alpha S_0}{V} + 2\mu S_0 + 3\alpha S_0 - \frac{S_0}{S}(\mu S_0 + \alpha S_0) \\
&\quad - \alpha\frac{V_0 S}{V} - (\mu+\gamma-\beta S_0(1+\nu I) - \beta_1 V_0) I \\
&\le -\mu S_0\left(\frac{S}{S_0} + \frac{S_0}{S} - 2\right) - \alpha S_0\left(\frac{V}{V_0} + \frac{SV_0}{S_0 V} + \frac{S_0}{S} - 3\right) \\
&\quad - (\mu+\gamma-\beta S_0(1+\nu I) - \beta_1 V_0(1+\nu I)) I \\
&= -\mu S_0\left(\frac{S}{S_0} + \frac{S_0}{S} - 2\right) - \alpha S_0\left(\frac{V}{V_0} + \frac{SV_0}{S_0 V} + \frac{S_0}{S} - 3\right) \\
&\quad - (\mu+\gamma-(\beta S_0 + \beta_1 V_0)(1+\nu I)) I \\
&= -\mu S_0\left(\frac{S}{S_0} + \frac{S_0}{S} - 2\right) - \alpha S_0\left(\frac{V}{V_0} + \frac{SV_0}{S_0 V} + \frac{S_0}{S} - 3\right) \\
&\quad - (\mu+\gamma)(1-R_{01}) I
\end{aligned}$$

Since all the model parameters are non-negative, it follows that $L' \le 0$ for all $S,V,I > 0$ if $R_{01} \le 1$ and $L' = 0$ if and only if $S = S_0$, $V = V_0$ and $I = 0$. Now using LaSalle's invariance principle (J.P. LaSalle, 1967) of Lyapunov method, the limit set of each solution is contained in the largest compact invariant set $\{(S,V,I) \in \Gamma : L'(S,V,I) = 0\}$, which is the singleton $\{E_0\}$. Therefore, every solution that starts in Γ approaches E_0 as $t \to \infty$. This completes the proof.

Theorem 4. If $R_0 > 1$, then the endemic equilibrium E^* of system (3) is locally asymptotically stable in Γ, provided $a_1 a_2 - a_3 > 0$, where a_1, a_2 and a_3 are presented in the proof.

Proof. The local stability for E^* is governed by the matrix is

$$J(E^*) = \begin{bmatrix} -\frac{\mu}{S^*} & 0 & -\beta S^*(1+2\nu I^*) \\ \alpha & -\frac{\alpha S^*}{V^*} & -\beta_1 V^* \\ \beta I^*(1+\nu I^*) & \beta_1 I^* & \nu\beta S^* I^* \end{bmatrix}$$

The characteristic equation of $J(E^*)$ is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$a_1 = \frac{\mu}{S^*} + \frac{\alpha S^*}{V^*} - \nu\beta S^* I^*,$$

$$\begin{aligned}
a_2 &= \frac{\alpha\mu}{V^*} + \beta_1^2 V^* I^* + \beta^2 S^* I^*(1+2\nu I^*) \\
&\quad + \nu\beta^2 S^* I^*(1+2\nu I^*) - \nu\beta S^* I^*\left(\frac{\mu}{S^*} + \frac{\alpha S^*}{V^*}\right),
\end{aligned}$$

$$\begin{aligned}
a_3 &= \alpha\beta\beta_1 S^* I^*(1+2\nu I^*) + \beta^2\alpha\frac{S^{*2} I^*}{V^*}(1+\nu I^*)(1+2\nu I^*) \\
&\quad + \beta_1^2\mu\frac{V^* I^*}{S^*} > 0
\end{aligned}$$

It is seen that

$$\begin{aligned}
a_1 &= \frac{\mu}{S^*} + \frac{\alpha S^*}{V^*} - \nu\beta S^* I^* \\
&= \beta I^* \left(1 + \nu I^*\right) + \beta_1 I^* + \left(\mu + \gamma_1\right) + \left(\mu + \alpha\right) - \nu\beta S^* I^* \\
&\geq \beta I^* + \nu\beta I^{*2} + \beta_1 I^* + \left(\mu + \gamma_1\right) + \left(\mu + \alpha\right) - \beta I^* \\
&= \nu\beta I^{*2} + \beta_1 I^* + \left(\mu + \gamma_1\right) + \left(\mu + \alpha\right) \\
&\geq \nu\beta_1 I^{*2} + \beta_1 I^* + \left(\mu + \gamma_1\right) + \left(\mu + \alpha\right) \\
&= \beta_1 I^* \left(1 + \nu I^*\right) + \left(\mu + \gamma_1\right) + \left(\mu + \alpha\right) > 0.
\end{aligned}$$

We also get

$$\begin{aligned}
&a_1 a_2 - a_3 \\
&= \left(\frac{\mu}{S^*} + \frac{\alpha S^*}{V^*} - \nu\beta S^* I^* \right) \left(\frac{\alpha\mu}{V^*} + \beta_1^2 V^* I^* \right. \\
&\quad \left. + \beta^2 S^* I^* \left(1 + \nu\right)\left(1 + 2\nu I^*\right) - \nu\beta S^* I^* \left(\frac{\mu}{S^*} + \frac{\alpha S^*}{V^*} \right) \right) \\
&\quad - \left(\beta\alpha\left(\mu + \gamma\right) \frac{S^* I^* \left(1 + 2\nu I^*\right)}{V^*} + \beta_1^2 \mu \frac{V^* I^*}{S^*} \right) \\
&= \left(\frac{\mu}{S^*} + \frac{\alpha S^*}{V^*} - \nu\beta S^* I^* \right) \left(\frac{\alpha\mu}{V^*} + \beta_1^2 V^* I^* \right. \\
&\quad \left. + \beta^2 S^* I^* \left(1 + \nu\right)\left(1 + 2\nu I^*\right) \right) - \nu\beta S^* I^* \left(\frac{\mu}{S^*} + \frac{\alpha S^*}{V^*} \right) \\
&\quad \times \left(\frac{\mu}{S^*} + \frac{\alpha S^*}{V^*} - \nu\beta S^* I^* \right) - \beta\alpha\left(\mu + \gamma\right) \frac{S^* I^* \left(1 + 2\nu I^*\right)}{V^*} \\
&\quad - \beta_1^2 \mu \frac{V^* I^*}{S^*}.
\end{aligned}$$

According to the Routh-Hurwitz criterion, the endemic equilibrium E^* is locally asymptotically stable in Γ if $a_1 a_2 - a_3 > 0$. This completes the proof.

4. Numerical simulations

To investigate the dynamical behavior of the model, the system (3) is integrated numerically with parameters values: $\mu = 0.01,\ \gamma = 1/5,\ \gamma_1 = 1/7,$ α=1 and $\nu = 0.5$ with various β and β_1.

With above parameter values $\beta = 4$ and $\beta_1 = 1.25$, the value of the basic reproduction number is $R_0 = 0.51297$, which yields $R_{01} = 0.76946 < 1$. Therefor, the system (3) has a disease-free equilibrium $E_0\left(0.00990, 0.06520, 0\right)$ and the solutions of the model (3) in the Figure 1 converge monotonically to E_0, as guaranteed by Theorem 1 and 2.

With above parameter values $\beta = 8$ and $\beta_1 = 6.5$, the value of the basic reproduction number is $R_0 = 2.38205 > 1$ and the system has a endemic equilibrium $E^*\left(0.00793, 0.02240, 0.03094\right)$. Figure 2 shows all solutions of the model (3) converge to E^* of the model (3) is locally asymptotically stable that is the disease persists.

For parameter values used: $\mu = 0.8,$ $\beta = 3.96,\ \beta_1 = 2.0,\ \gamma = 0.5,\ \gamma_1 = 0.4,\ \alpha = 1.0$ for each value ν; $\nu = 0,\ 0.5,\ 1.0$ and given by $R_0 = 1.9236467 > 1$. Thus, the system (3) has a uniqueequilibrium

$E^*\left(0.2580750, 0.1390115,\ 0.3282510\right)$,

$E^*\left(0.2272379, 0.1174843, 0.367099\right)$,

and $E^*\left(0.1984950, 0.0990708, 0.4017839\right)$,

where $\nu = 0,\ 0.5$ and $\nu = 1.0,$ respectively. Figure 3 shows that the solution are locally asymptotically stable as guaranteed by Theorem 1 and 4. Moreover, it is found, see Figure 3, that even R_0 does not depend on ν but the density of infected population increases as ν increases. This verifies that non-linear incidence rate has effected to the density of infectious populations.

5. Conclusion

In this paper we study the global stability of the *SVIR* model with nonlinear incidence rate $\beta SI(1 + \nu I)$, where $\beta > 0$ and $0 < \nu \leq 1$. Our result show that the model has the reproduction number is similar to the model, which is proposed by Lie et al. (2008). (i.e., the reproduction number of the study system (3) does not depend on the parameter ν in the nonlinear incidence function). The following results were obtained:

(i) The model has a locally and globally asymptotically stable whenever $R_{01} \le 1$.

(ii) A unique endemic of the model has a locally asymptotically stable whenever the basic reproduction number is above the unity.

(iii) In the numerical simulation of this model, the parameters ν and β_1 are effect to the infectious population and an endemic equilibrium that those parameters measure the effects of sociological, psychological or other mechanisms disease.

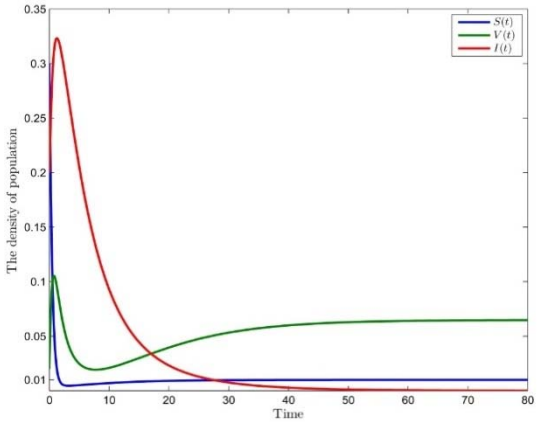

Figure 1 Time series plots for the system (3) with parameter values used:
$\mu = 0.01,\ \beta = 4,\ \beta_1 = 1.25,$
$\gamma = 1/5,\ \gamma_1 = 1/7$ and $\nu = 0.5$.

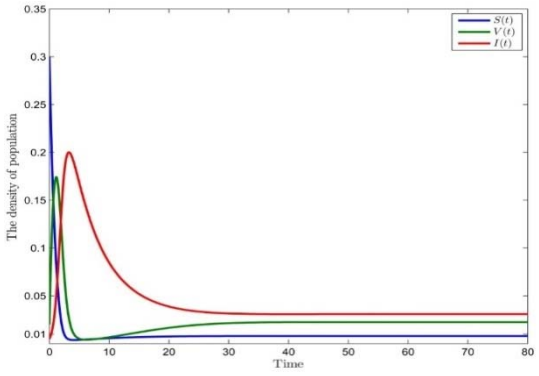

Figure 2 Time series plots for the system (3) with parameter values used:
$\mu = 0.01,\ \beta = 8,\ \beta_1 = 6.5,$
$\gamma = 1/5,\ \gamma_1 = 1/7$ and $\nu = 0.5$.

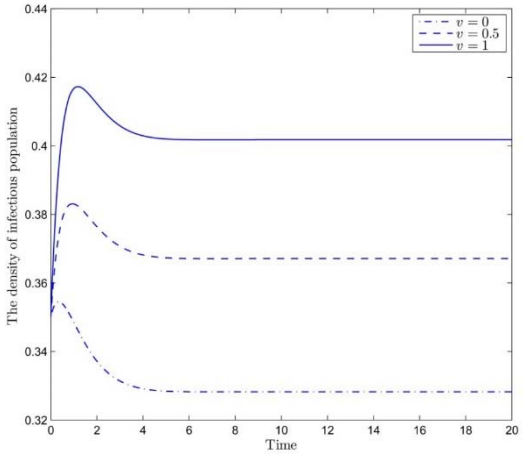

Figure 3 Timeseries plots for the density of infectiousindividuals $I(t)$ of the system (3) with parameter values used $\mu = 0.8,$ $\beta = 3.96, \beta_1 = 2.0, \gamma = 0.5,$ $\gamma_1 = 0.4, \alpha = 1.0$ to compare the different parameters $\nu; \nu = 0,\ 0.5$ and $\nu = 1.0$, respectively.

Acknowledgements

This study was granted by faculty of Science and supported byScience Achievement Scholarship of Thailand (SAST), King Mongkut's University of Technology Thonburi, Thailand.

References

Alexander M.E. and Moghadas S.M. 2004.Periodicity in an epidemic model with a generalizednon-linear incidence.Math.Biosci. 189: 75-96.

Jin Y., Wang W. and Xiao S. 2007.An SIRS models with a nonlinear incidence rate. Chaos, Solitons and Fractals. 34: 1482-1497.

Kermarck W.O. and Mckendrick A.G. 1927. A contributions to the mathematical theory of epidemics. Proceedings of the Royal Society of London. 115: 700-721.

Kribs-Zaleta C.M. and Velasco-Hernández J.X. 2000. A simple vaccination model with multiple endemic states. Mathematical Biosciences. 164: 183-201.

LaSalle J.P. 1967. The stability of Dynamical Systems, CBMS-NSF Regional Conference Service in Application Mathemetics 25, SIAM, Philadelphia.

Liu X., Takeuchi Y. and Iwami S. 2008. SVIR epidemic models with vaccination strategies. Journal of Theoretical Biology. 253: 1-11.

van der Driessche P. and Watmough J. 2000. A simple SIS epidemic model with a backward bifurcation, Journal of Mathematical Biology. 40: 525-540.

van der Driessche P. and Watmough J. 2003. Epidemic solution and endemic catastrophes.Fields Institute Communications. 36: 247-257.

Zhou X. and Cui J. 2011. Analysis of stability and bifurcation for and SEIV epidemic model with vaccination and nonlinear incidence rate. Nonlinear Dynamics. 63: 639-653.