# On the Principal Bi-Ideals in the Direct Product of Two Semigroups

# ไบ-ไอดีลมุขสำคัญในผลคูณตรงของสองกึ่งกรุป

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## ABSTRACT

Let $S$ and $T$ be any two semigroups. For an element $x$ of $S$, the symbols $B(x)$ and $B_x$ stand for a bi-ideal of $S$ generated by $x$, and $\mathcal{B}$-class of $S$ containing $x$, respectively. Let $s \in S$ and $t \in T$. In this paper, we characterize when 1) $B((s,t)) = B(s) \times B(t)$; 2) $B_{(s,t)} = B_s \times B_t$ and 3) $B_{(s,t)}$ is a maximal $\mathcal{B}$-class in $S \times T$.

## บทคัดย่อ

ให้ $S$ และ $T$ เป็นกึ่งกรุปใดๆ สำหรับแต่ละสมาชิก $x$ ของ $S$ เรากำหนดให้สัญลักษณ์ $B(x)$ และสัญลักษณ์ $B_x$ คือไบ-ไอดีลมุขสำคัญของ $S$ ก่อกำเนิดโดยสมาชิก $x$ และ $\mathcal{B}$-คลาส ของ $S$ ที่บรรจุสมาชิก $x$ ตามลำดับ ให้ $s \in S, t \in T$ ในบทความนี้ เราจะทำการอธิบายลักษณะเมื่อ 1) $B((s,t)) = B(s) \times B(t)$; 2) $B_{(s,t)} = B_s \times B_t$ และ 3) $B_{(s,t)}$ เป็น $\mathcal{B}$-คลาสสูงสุดเฉพาะที่

**Keywords:** Direct product, Semigroup, Bi-ideal

**คำสำคัญ:** ผลคูณตรง กึ่งกรุป ไบ-ไอดีล ไบ-ไอดีลมุขสำคัญ $\mathcal{B}$-คลาสสูงสุดเฉพาะที่

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## Introduction

Let $S$ and $T$ be semigroups. Then the Cartesian product $S \times T$ becomes a semigroup if we define

$$(s,t)(s',t') = (ss',tt')$$

for every $(s,t),(s',t') \in S \times T$.

We refer to this semigroup as the *direct product* of $S$ and $T$ (Howie, 1995). The direct product is one of the interesting topics in semigroup. Frabici (1991) characterized the principal one-side ideal in the direct product of two semigroups. In this article we extend these ideas to the principal bi-ideals which is defined by Good, Hughes (1952). The necessary tool to prove theorems in this article is the projection map. Let $S$ and $T$ be any two semigroups, the projection map $\pi_S: S \times T \to S$ is defined by $(s,t)\pi_S = s$ for all $(s,t) \in S \times T$. The projection map $\pi_T: S \times T \to T$ is defined analogously.

From now, we write $AB$ to mean $\{ ab \mid a \in A, b \in B \}$ for every subset $A, B$ of a semigroup $S$.

## The direct product of principal bi-ideals

The first definition is applied from ideas of Lajos (1961).

**Definition 2.1.** Let $C$ be a subsemigroup a semigroup $S$. We say that $C$ is a *bi-ideal* of $S$ if $CSC \subseteq C$. For a subset $A$ of $S$, the smallest bi-ideal of $S$ containing $A$ is called *the bi-ideal of S generated by* $A$, and denoted by $B(A)$. In particular, if $A = \{a\}$, $B(A)$ is called *the principal bi-ideal of S generated by* $a$, and replaced by $B(a)$.

**Example 1.** Let $S = \{a,b,c,d,e\}$ be a semigroup with the operation $\cdot$ defined by

| $\cdot$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $d$ | $c$ | $a$ |
| $b$ | $a$ | $b$ | $c$ | $d$ | $b$ |
| $c$ | $d$ | $c$ | $d$ | $c$ | $c$ |
| $d$ | $c$ | $d$ | $c$ | $d$ | $d$ |
| $e$ | $a$ | $b$ | $c$ | $d$ | $e$ |

Let $A = \{c,d\}$. Then $A$ is a subsemigroup of $S$ because $AA = \{c,d\} = A$. Moreover, $A$ is a bi-ideal of $S$ because $ASA = \{c,d\} = A$.

Let $B = \{a,b\}$. Since $BB = B$, it follows that $B$ is a subsemigroup of $S$. But $B$ is not bi-ideal because $acb = d \notin B$.

Let $C = \{a,b,c,d\}$. It can be observed that $C$ is a bi-ideal of $S$. Moreover, $C$ is the principal bi-ideal of $S$ generated by $a$.

**Lemma 2.2.** (Lajos,1960) Let $a$ be an element of a semigroup $S$. Then $B(a) = \{a,a^2\} \cup aSa$.

**Lemma 2.3.** Let $s$ be an element of a semigroup $S$. Then $B(s) = sSs$ if and only if $s \in sSs$.

*Proof.* Suppose that $B(s) = sSs$. Then $s \in B(s) \subseteq sSs$. Conversely, we assume that $s \in sSs$. It is clear that $sSs$ is a bi-ideal of $S$. Therefore, $B(s) \subseteq sSs$. The reverse inclusion is obtained directly from Lemma 2.2. Thus $B(s) = sSs$.

**Lemma 2.4.** Let $s$ be an element of a semigroup $S$. Then $B(s)SB(s) = sSs$.

*Proof.* This follows by

$$\begin{aligned} sSs &\subseteq B(s)SB(s) \\ &= (\{s,s^2\} \cup sSs)S(\{s,s^2\} \cup sSs) \\ &= sSs. \end{aligned}$$

**Lemma 2.5.** Let $S$ and $T$ be any two semigroups, and let $s \in S, t \in T$. Then

$$B((s,t)) = \{(s,t),(s,t)^2\} \cup (sSs \times tTt).$$

*Proof.* This follows by

$$\begin{aligned} B\big((s,t)\big) &= \{(s,t),(s,t)^2\} \cup (s,t)(S\times T)(s,t) \\ &= \{(s,t),(s,t)^2\} \cup (sSs \times tTt). \end{aligned}$$

**Lemma 2.6.** Let $S$ and $T$ be any two semigroups, and let $s \in S, t \in T$. Then $B\big((s,t)\big) \subseteq B(s) \times B(t)$.

*Proof.* This follows by Lemma 2.5.

**Theorem 2.7.** Let $S$ and $T$ be any two semigroups, and let $s \in S, t \in T$. Then $B\big((s,t)\big) = B(s) \times B(t)$ if and only if at least one of the following conditions holds:

(1) $sSs = \{s\}$.

(2) $tTt = \{t\}$.

(3) $s \in sSs$ and $t \in tTt$.

*Proof.* Assume first that $B\big((s,t)\big) = B(s) \times B(t)$. If $s \notin sSs$, then $s \neq s^k$ for all $k \in \{2,3,\dots\}$. Since $B\big((s,t)\big) = B(s) \times B(t)$, we obtain

$$\{s\} \times tTt = \{(s,t)\}.$$

Therefore, $tTt = \{t\}$. Similarly, if $t \notin tTt$, then $sSs = \{s\}$. Conversely, we assume that (1),(2) or (3) holds.

Case 1: $sSs = \{s\}$. Then, by Lemma 2.3, $B(s) = sSs = \{s\}$. By Lemma 2.2,

$$\begin{aligned} B(s) \times B(t) &= \{s\} \times B(t) \\ &= \{(s,t),(s,t^2)\} \cup (\{s\} \times tTt) \\ &= \{(s,t),(s^2,t^2)\} \cup (sSs \times tTt) \\ &= \{(s,t),(s,t)^2\} \cup (sSs \times tTt) \\ &= B\big((s,t)\big). \end{aligned}$$

Case 2: $tTt = \{t\}$. We proceed similarly Case 1.

Case 3: $s \in sSs$ and $t \in tTt$. Then, by Lemma 2.3, $B(s) \times B(t) = sSs \times tTt$. By Lemma 2.6 and Lemma 2.3,

$$\begin{aligned} B\big((s,t)\big) &\subseteq B(s) \times B(t) \\ &= sSs \times tTt \\ &\subseteq B\big((s,t)\big). \end{aligned}$$

Therefore, $B\big((s,t)\big) = B(s) \times B(t)$.

## The direct product of $\mathcal{B}$-classes

In the above section, we characterize the direct product of two principal bi-ideals. Now, we move to some equivalence class in $S \times T$. First, we consider a relation $\mathcal{B}$, which is defied by Kapp (1969), on any semigroup $S$ by for $a, b \in S$,

$$a\mathcal{B}b \text{ if } B(a) = B(b).$$

It is clear that $\mathcal{B}$ is an equivalence relation on $S$. We let $B_x$ be an $\mathcal{B}$-class of $S$ containing $x$ for every $x \in S$.

**Lemma 3.1.** Let $s$ be an element of a semigroup $S$. If $B_s \cap sSs \neq \emptyset$, then $B_s \subseteq sSs$.

*Proof.* Suppose that $B_s \cap sSs \neq \emptyset$. Then there exists $u \in B_s \cap sSs$. Thus

$$\begin{aligned} s &\in B(s) \\ &= B(u) \\ &\subseteq B(sSs) \\ &= sSs. \end{aligned}$$

By Lemma 2.3, $B(s) = sSs$. Let $v \in B_s$. Then

$$\begin{aligned} v &\in B(v) \\ &= B(s) \\ &= sSs. \end{aligned}$$

Therefore, $B_s \subseteq sSs$.

**Lemma 3.2.** Let $s$ be an element of a semigroup $S$. If the cadinality $|B_s| > 1$, then $B_s \subseteq sSs$.

*Proof.* Assume that $|B_s| > 1$. Then there exists $u \in B_s$ such that $u \neq s$. We have

$$\begin{aligned} u &\in B(u) \\ &= B(s) \\ &= \{s, s^2\} \cup sSs. \end{aligned}$$

Now, we have two cases to be considered:

Case 1: $u \in sSs$. Then $B_s \cap sSs \neq \emptyset$. By Lemma 3.1, $B_s \subseteq sSs$.

Case 2: $u = s^2$. Then $s \in B(u) = B(s^2)$. This leads to $s \in sSs$ and $B_s \cap sSs \neq \emptyset$. By Lemma 3.1, $B_s \subseteq sSs$.

From both cases, we obtain $B_s \subseteq sSs$.

**Theorem 3.3.** Let $S$ and $T$ be any two semigroups, and let $s \in S, t \in T$. Then the following conditions holds:

(1) $B_{(s,t)} \subseteq B_s \times B_t$.

(2) If $B_{(s,t)} \subset B_s \times B_t$, then $B_s \times B_t$ is the union of at least two $\mathcal{B}$-classes in $S \times T$.

*Proof.* To prove (1), let $(u, v) \in B_{(s,t)}$. Then

$$B(u, v) = B(s, t).$$

Since

$$B(s) = B\big((s,t)\big)\pi_S = B\big((u,v)\big)\pi_S = B(u)$$

and

$$B(t) = B((s,t))\pi_T = B\big((u,v)\big)\pi_T = B(v),$$

we have $(u, v) \in B_s \times B_t$.

To prove (2), suppose that $B_{(s,t)} \subset B_s \times B_t$. We let $(u, v) \in (B_s \times B_t) \setminus B_{(s,t)}$. Then

$$B(u) = B(s) \text{ and } B(v) = B(t).$$

Thus $B_{(u,v)} \subseteq B_u \times B_v = B_s \times B_t$. Hence, $B_{(s,t)}$ and $B_{(u,v)}$ are different classes of $S \times T$ contained in $B_s \times B_t$.

**Corollary 3.4.** Let $S$ and $T$ be any two semigroups, and let $s \in S, t \in T$. If $B_s = \{s\}$ and $B_t = \{t\}$, then

$$B_{(s,t)} = B_s \times B_t = \{(s,t)\}.$$

**Theorem 3.5.** Let $S$ and $T$ be any two semigroups, and let $s \in S, t \in T$. Then $B_s \times B_t = B_{(s,t)}$ if and only if at least one of the following conditions holds:

(1) $B_s = \{s\}$ and $B_t = \{t\}$.

(2) $s \in sSs$ and $t \in tTt$.

*Proof.* Assume that $B_s \times B_t = B_{(s,t)}$. If $|B_{(s,t)}| = 1$, then $B_s \times B_t = B_{(s,t)} = \{(s,t)\}$.

Therefore, $B_s = \{s\}$ and $B_t = \{t\}$. If $\left|B_{(s,t)}\right| > 1$, it follows by Lemma 3.2 that

$$\begin{aligned}(s,t) &\in B_{(s,t)} \\ &\subseteq (s,t)(S \times T)(s,t) \\ &= sSs \times tTt.\end{aligned}$$

Thus $s \in sSs$ and $t \in tTt$.

Conversely, suppose that (1) or (2) holds. If (1) holds, then, it is obtained by Corollary 3.4 that

$$B_{(s,t)} = B_s \times B_t.$$

Finally, we consider when (2) holds. By Theorem 2.7, we have

$$B(s,t) = B(s) \times B(t).$$

Since $B_{(s,t)} \subseteq B_s \times B_t$, we remain to prove that

$$B_s \times B_t \subseteq B_{(s,t)}.$$

Let $(u, v) \in B_s \times B_t$. If $(u, v) = (s, t)$, then we have immediately that $(u, v) \in B_{(s,t)}$. Now, suppose that $(u, v) \neq (s, t)$. Then we have two cases to be considered:

Case 1: If $u \neq s$, then we have, by Lemma 3.2, $u \in B_u \subseteq uSu$ because $s, u \in B_s = B_u$.

$\alpha$) If $v = t$, then $v \in vTv$. By Theorem 2.7,

$$\begin{aligned}B\big((u,v)\big) &= B(u) \times B(v) \\ &= B(s) \times B(t) \\ &= B((s,t)).\end{aligned}$$

Thus $(u, v) \in B_{(s,t)}$.

$\beta$) If $v \neq t$, then $v \in B_v \subseteq vTv$ by Lemma 3.2. As $\alpha$), we have $(u, v) \in B_{(s,t)}$ and thus

$$B_s \times B_t = B_{(s,t)}.$$

Case 2: If $u = s$ and $v \neq t$, then we prove analogously as case $\alpha$).

This completes the proof.

**Corollary 3.6.** Let $S$ and $T$ be any two semigroups, and let $s \in S, t \in T$. If $|B_s| > 1$ and $|B_t| > 1$, then

$$B_{(s,t)} = B_s \times B_t.$$

**Corollary 3.7.** Let $S$ and $T$ be any two semigroups, and let $s \in S, t \in T$. If $B_s \times B_t$ is the union of at least two $\mathcal{B}$-classes, then ($|B_s| > 1$ and $B_t = \{t\}$) or ($B_s = \{s\}$ and $|B_t| > 1$).

**Theorem 3.8.** Let $S$ and $T$ be any two semigroups, and let $s \in S, t \in T$. Then $B_s \times B_t$ is the union of at least two $\mathcal{B}$-classes if and only if either

$$|B_s| > 1, B_t = \{t\}, t \notin tTt$$

or

$$|B_t| > 1, B_s = \{s\}, s \notin sSs.$$

*Proof.* Assume that $B_s \times B_t$ is the union of at least two classes. By above Corollary, it follows that ($|B_s| > 1$ and $B_t = \{t\}$) or ($B_s = \{s\}$ and $|B_t| > 1$).

Case 1: $|B_s| > 1, B_t = \{t\}$. Then $t \notin tTt$ because otherwise, $s \in sSs$ and $t \in tTt$ imply $B_s \times B_t = B_{(s,t)}$.

Case 2: $|B_t| > 1, B_s = \{s\}$. This can be proved analogously, and hence $s \notin sSs$.

Conversely, it is sufficient to consider the case

$$|B_s| > 1, B_t = \{t\}, t \notin tTt.$$

Let $u \in B_s$ such that $u \neq s$. Then $(u, t) \in B_s \times B_t$.

Since $t \notin tTt$, we have $(s, t) \notin sSs \times tTt$.

By Lemma 3.2,

$$(u, t) \notin \{(s, t)\} = B_{(s,t)}.$$

Thus $B_s \times B_t$ contains at least two $\mathcal{B}$-classes including $B_{(s,t)}$ and $B_{(u,t)}$.

## The maximal $\mathcal{B}$-classes in $S \times T$

In the previous sections, we can see that the properties $s \in sSs, t \in tTt$ are used to prove many theorem of this article. In the final section, we consider the maximality of $\mathcal{B}$-class in $S \times T$ such that element $s \in S$ and $t \in T$ have a property $(s, t) \in sSs \times tTt$.

**Definition 4.1.** Let $s$ be an element of a semigroup $S$. A $\mathcal{B}$-class $B_s$ is *maximal* if there is no element $u \in S$ such that $B(s) \subset B(u)$.

**Lemma 4.2.** Let $S$ and $T$ be any two semigroups, and let $s \in S, t \in T$ be such that $(s, t) \in sSs \times tTt$. Then for each $u \in S, v \in T$, $B\big((s,t)\big) \subseteq B\big((u,v)\big)$ if an only if $B(s) \subseteq B(u)$ and $B(t) \subseteq B(v)$.

*Proof.* Assume that $B\big((s,t)\big) \subseteq B\big((u,v)\big)$. Then

$$\begin{aligned} B(s) &= B\big((s,t)\big)\pi_S \\ &\subseteq B\big((u,v)\big)\pi_S \\ &= B(u) \end{aligned}$$

and

$$\begin{aligned} B(t) &= B\big((s,t)\big)\pi_T \\ &\subseteq B\big((u,v)\big)\pi_T \\ &= B(v). \end{aligned}$$

Hence $B(s) \subseteq B(u)$ and $B(t) \subseteq B(v)$.

Conversely, assume that $B(s) \subseteq B(u)$ and $B(t) \subseteq B(v)$. Since $(s,t) \in sSs \times tTt$, it follows by Theorem 2.7 and Lemma 2.3 that

$$B\big((s,t)\big) = B(s) \times B(t) = sSs \times tTt.$$

Thus

$$\begin{aligned} B\big((s,t)\big) &= sSs \times tTt \\ &\subseteq B(u)SB(u) \times B(v)TB(v) \\ &= uSu \times vSv \\ &\subseteq B((u,v)). \end{aligned}$$

**Theorem 4.3.** Let $S$ and $T$ be any two semigroups, and let $s \in S, t \in T$ such that $(s,t) \in sSs \times tTt$. Then $B_{(s,t)}$ is a maximal $\mathcal{B}$-class in $S \times T$ if and only if $B_s$ and $B_t$ are maximal $\mathcal{B}$-classes in $S$ and in $T$, respectively.

*Proof.* Suppose that $B_{(s,t)}$ is a maximal $\mathcal{B}$-class in $S \times T$. If $B_s$ is not maximal $\mathcal{B}$-class in $S$, then there exists $u \in S$ such that $B(s) \subset B(u)$. By Lemma 4.2, we have

$$B\big((s,t)\big) \subseteq B\big((u,t)\big).$$

But

$$(u,t) \notin B(s) \times B(t) = B\big((s,t)\big).$$

Thus $B\big((s,t)\big) \subset B\big((u,t)\big)$, a contradiction. Therefore, $B_s$ is a maximal $\mathcal{B}$-class in $S$. Similarly, we can prove that $B_t$ is a maximal $\mathcal{B}$-class in $T$. Conversely, we assume that $B_s$ and $B_t$ are maximal $\mathcal{B}$-classes in $S$ and in $T$, respectively. If $B_{(s,t)}$ is not maximal, then there exists $(u,v) \in S \times T$ such that $B\big((s,t)\big) \subset B\big((u,v)\big)$. Thus

$$\begin{aligned} B(s) \times B(t) &= B\big((s,t)\big) \\ &\subset B\big((u,v)\big) \\ &\subseteq B(u) \times B(v), \end{aligned}$$

a contradiction. Hence $B_{(s,t)}$ is a maximal $\mathcal{B}$-class of $S \times T$.

## Conclusion

Although the direct product of the principal bi-ideals of $S$ and $T$ is not the principal bi-ideal on $S \times T$ in general, but we can determine the properties of generators to affect equality of them. At the same time, these properties affect the direct product of $\mathcal{B}$-classes of $S$ and $T$. Moreover, these properties preserve an inclusion of maximal $\mathcal{B}$-class as well.

## Acknowledgements

This work has been supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

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