A new method for equilibrium problems and a finite family of pseudocontractive mapping
วิธีการใหม่สำหรับปัญหาเชิงดุลยภาพและการส่งของวงศ์จำกัดแบบหดเทียม

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ABSTRACT

The purpose of this paper is to prove the strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed point of a finite family of κ_i-strictly pseudocontractive mapping (i.e, there exist $\kappa_i \in [0,1)$ such that $\|Tx-Ty\|^2 \le \|x-y\|^2 + \kappa_i \|(I-T)x-(I-T)y\|^2$ $\forall x, y \in C$ and $i = 1, 2, ..., N$) in framework of Hilbert spaces by using different methods. Our main theorem improve and modify some previously proposed results.

บทคัดย่อ

วัตถุประสงค์ของงานวิจัยฉบับนี้คือพิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มสำหรับการหาสมาชิกร่วมของเซตของผลเฉลยของปัญหาเชิงดุลยภาพและเซตของจุดตรึงของการส่งวงศ์จำกัดของการส่ง κ_i หดเทียมโดยแท้ (กล่าวคือ มี $\kappa_i \in [0,1)$ ซึ่งทำให้ $\|Tx-Ty\|^2 \le \|x-y\|^2 + \kappa_i \|(I-T)x-(I-T)y\|^2$, สำหรับทุกๆ $x, y \in C$ และ $i = 1, 2, ..., N$) ในปริภูมิฮิลเบิร์ต โดยใช้วิธีที่แตกต่างจากงานวิจัยอื่นๆ ซึ่งผลการศึกษาของงานวิจัยฉบับนี้จะพัฒนาและปรับปรุงผลการศึกษาของงานวิจัยก่อนหน้านี้ได้

Keywords: Strictly pseudocontractive mapping, Equilibrium problem, Fixed point

คำสำคัญ: การส่งหดเทียมโดยแท้ ปัญหาดุลยภาพ จุดตรึง

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Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let P_C be the projection of H onto C. A mapping T of H into itself is called nonexpansive if $\|Tx-Ty\| \le \|x-y\|$ for all $x, y \in H$. The set of fixed points of T is denoted by $F(T)$ $(i.e, F(T)=\{x \in H : Tx = x\})$. A mapping $T: C \to C$ is said to be a κ-strictly pseudo contraction mapping, if there exist $\kappa \in [0,1)$ such that $\|Tx-Ty\|^2 \le \|x-y\|^2 + \kappa\|(I-T)x-(I-T)y\|^2$, $\forall x, y \in C.$

Let $F: C \times C \to R$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points, i.e. the set

$$EP(F) = \{x \in C : F(x,y) \ge 0, \forall y \in C\}. \tag{1.1}$$

The problem is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others, see (Blum, Oettli, 1994; Combettes, Hirstoaga, 2005; Takahashi, Takahashi, 2007).

Takahashi, Takahashi (2007) introduced viscosity approximation method for finding a common element of the set of solutions of problem (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space. They defined the sequences $\{x_n\}$ and $\{u_n\}$ as follows: Let $x_1 \in H$ and $f: H \to H$ be a contraction mapping with $\alpha \in (0,1)$, $\{x_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \ge 0, & \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tu_n, & \forall n \in N, \end{cases} \tag{1.2}$$

They proved strong convergence theorem of sequence $\{x_n\}$ generated by (1.2) to $z \in F(T) \bigcap EP(F)$ where $z = P_{F(T)\bigcap EP(F)} f(z)$ under certain appropriate conditions imposed on the sequences $\{\alpha_n\}, \{r_n\}$ and bifunction F.

In this paper, motivated by (1.2), we introduce a new algorithm as follows: for $u \in C$ and the sequence $\{x_n\}$ generated by

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y-u_n, u_n - (\beta_n Tx_n + (1-\beta_n)x_n)\rangle \ge 0, & \forall y \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, & \forall n \ge 1, \end{cases} \tag{1.3}$$

where $F: C \times C \to R$ is a bifunction and $T_i : C \to C$ is κ_i-strictly pseudocontractive mapping. Under suitable conditions of parameters $\{\alpha_n\}, \{\beta_n\}, \{r_n\}$, we prove strong convergence theorem for finding a common element of the set of equilibrium problem of finite family of κ_i-strictly pseudocontractive mapping in Hilbert spaces.

Objective of the study

To introduce a new algorithm for finding a common element of the set of solutions of equilibrium problem and the set of fixed point of a finite family of κ_i-strictly pseudocontractive mapping in framework of Hilbert spaces.

Preliminaries

In this section, we give some useful lemmas and definitions that will be needed for our main result.

Definition 2.1.[5] Let E be a real Banach space and D be a closed subset of E. A mapping $T : D \to D$ is said to be demi-closed at the origin if, for any sequence $\{x_n\}$ in D, the conditions $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly imply $Tx_0 = 0$.

Lemma 2.1.[7] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$s_{n+1} = (1-\alpha_n)s_n + \beta_n, \quad \forall n \geq 0$ where $\{\alpha_n\}, \{\beta_n\}$ satisfy conditions

(1) $\{\alpha_n\} \subset (0,1), \ \sum_{n=1}^{\infty} \alpha_n = \infty,$

(2) $\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \leq 0.$

Then $\lim_{n\to\infty} s_n = 0.$

Let C be closed convex subset of real Hilbert space H and P_C be the metric projection of H onto C i.e, for $x \in H, P_C x$ satisfies the property $\|x - P_C x\| = \min_{y\in C} \|x - y\|$.

The following lemma 2.2. characterizes the projection P_C.

Lemma 2.2.[5] Given $x \in H$ and $y \in C.$ Then $P_C x = y$ if and only if there holds the inequality $\langle x - y, y - z\rangle \geq 0 \quad \forall z \in C.$

Lemma 2.3.[2] Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.

Lemma 2.4.[8] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$s_{n+1} = (1-\alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$ where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty,$

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0 \ or \ \sum_{n=1}^{\infty} |\delta_n| < \infty.$

Then $\lim_{n\to\infty} s_n = 0.$

For solving the equilibrium problem for a bifunction $F : C \times C \to R,$ let us assume that F satisfy the following conditions:

(A1) $F(x, x) = 0 \quad \forall x \in C;$

(A2) F is monotone, $F(x, y) + F(y, x) \leq 0, \quad \forall x, y \in C;$

(A3) $\forall x,y,z\in C,\ \lim_{t\to 0^+} F(tz+(1-t)x,y)\le F(x,y);$

(A4) $\forall x\in C,\ y\mapsto F(x,y)$ is convex and lower semi continuous.

Lemma 2.5.[1] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C\times C$ into R satisfying (A1) - (A4). Let $r>0$ and $x\in H.$ Then, there exists $z\in C$ such that

$$F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle>0 \qquad (2.1)$$

for all $y\in C.$

Lemma 2.6.[3] Assume that $F:C\times C\to R$ satisfies (A1) – (A4). For $r>0$ and $x\in H,$ define a mapping $S_r:H\to C$ as follows:

$$S_r(x)=\left\{z\in C:F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\ge 0,\ \forall y\in C\right\}. \qquad (2.2)$$

for all $z\in H$ .Then, the following hold:

(1) S_r is single-valued;

(2) S_r is firmly nonexpansive i.e, $\|S_r(x)-S_r(y)\|^2\le\langle S_r(x)-S_r(y),x-y\rangle\ \ \forall x,y\in H;$

(3) $F(S_r)=EP(F);$

(4) $EP(F)$ is closed and convex.

Lemma 2.7.[9] In a real Hilbert spaces H ,there holds the following inequality $\|x+y\|^2\le\|x\|^2+2\langle y,x+y\rangle$ for all $x,y\in H.$

Lemma 2.8.[9] Let C be a nonempty closed convex subset of real Hilbert space H and $T:C\to C$ a κ - strict pseudo contraction. Define $S:C\to C$ by $Sx=\alpha x+(1-\alpha)Tx$ for each $x\in C.$ Then $\alpha\in[\kappa,1),$ S is nonexpansive such that $F(S)=F(T).$

Lemma 2.9.[4] Let C be a nonempty closed convex subset of strictly convex. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mapping of C into itself with $\bigcap_{i=1}^N F(T_i)\ne\varnothing$ and let $\alpha_j=(\alpha_1^j,\alpha_2^j,\alpha_3^j)\in I\times I\times I,$ $j=1,2,3,\ldots,N$, where $I=[0,1],$ $\alpha_1^j+\alpha_2^j+\alpha_3^j=1,\ \alpha_1^j\in(0,1)$ for all $j=1,2,3,\ldots,N-1,$ $\alpha_1^N\in(0,1],\ \alpha_2^j,\ \alpha_3^j\in[0,1)$ for all $j=1,2,3,\ldots,N.$ Let S be the mapping generated by $T_1,T_2,\ldots,T_N$ and $\alpha_1,\alpha_2,\ldots,\alpha_N$. Then $F(S)=\bigcap_{i=1}^N F(T_i).$

Main result

Theorem 3.1 Let C be a nonempty closed convex subset of a Hilbert space H . Let F be a bifunctions from $C\times C$ into R satisfying (A1) – (A4). Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i - strictly pseudocontractions with $\mathbf{F}=\bigcap_{i=1}^N F(T_i)\bigcap EP(F)\ne\varnothing.$ Define a mapping T_{κ_i} by $T_{\kappa_i}=\kappa_i x+(1-\kappa_i)T_i x,\quad \forall x\in C,$ $i\in\{1,2,\ldots,N\}.$ Let $\alpha_j=(\alpha_1^j,\alpha_2^j,\alpha_3^j)\in I\times I\times I,\ \ j=1,2,3,\ldots,N$, where $I=[0,1],$ $\alpha_1^j+\alpha_2^j+\alpha_3^j=1,\ \alpha_1^j\in(0,1)$ *for all* $j=1,2,3,\ldots,N-1,\ \alpha_1^N\in(0,1],\ \alpha_2^j,\ \alpha_3^j\in[0,1)$ *for all* $j=1,2,3,\ldots,N.$ Let S be the mapping generated by $T_{\kappa_1},T_{\kappa_2},\ldots,T_{\kappa_N}$ and $\alpha_1,\alpha_2,\ldots,\alpha_N$. Let $\{x_n\}$ be sequence generated by $x_1,u\in C:$

$$\begin{cases} F(u_n,y)+\dfrac{1}{r_n}\langle y-u_n,u_n-(\beta_n Sx_n+(1-\beta_n)x_n)\rangle \geq 0, & \forall y\in C, \\ x_{n+1}=\alpha_n u+(1-\alpha_n)u_n, & \forall n\geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\},\{\beta_n\}\subset(0,1)$ such that $r_n\in(a,b)$ and $\beta_n\in[c,d]\subset(0,1]$.

Assume that

$(i)\ \lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty,$

$(ii)\ \sum_{n=0}^{\infty}|r_{n+1}-r_n|,\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|,\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty.$

Then the sequence $\{x_n\}$ converge strongly to $z=P_{\mathbf{F}}u.$

Proof.**Step 1.** We shall show that the sequence $\{x_n\}$ is bounded.

Let $z\in F$. Since

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-(\beta_n Sx_n+(1-\beta_n)x_n)\rangle\geq 0,\ \ \forall y\in C, \tag{3.2}$$

by Lemma 2.6, it implies that $u_n=S_{r_n}(\beta_n Tx_n+(1-\beta_n)x_n)$ and $z\in F(S_{r_n})$.

Then, we have

$$\begin{aligned}\|x_{n+1}-z\| &\leq \alpha_n\|u-z\|+(1-\alpha_n)\|u_n-z\|\\ &=\alpha_n\|u-z\|+(1-\alpha_n)\|S_{r_n}(\beta_n Sx_n+(1-\beta_n)x_n)-z\|\\ &\leq\alpha_n\|u-z\|+(1-\alpha_n)\|(\beta_n Sx_n+(1-\beta_n)x_n)-z\|\\ &\leq\alpha_n\|u-z\|+(1-\alpha_n)(\beta_n\|Sx_n-z\|+(1-\beta_n)\|x_n-z\|)\\ &\leq\alpha_n\|u-z\|+(1-\alpha_n)(\beta_n\|x_n-z\|+(1-\beta_n)\|x_n-z\|)\\ &=\alpha_n\|u-z\|+(1-\alpha_n)\|x_n-z\|\\ &\leq\max\{\|u-z\|,\|x_n-z\|\}.\end{aligned} \tag{3.3}$$

By induction we can prove that $\{x_n\}$ is bounded, so are $\{Sx_n\}$ and $\{u_n\}$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$

Put $y_n=\beta_n Sx_n+(1-\beta_n)x_n.$

Consider

$$\begin{aligned}\|x_{n+1}-x_n\| &=\|\alpha_n u+(1-\alpha_n)u_n-\alpha_{n-1}u-(1-\alpha_{n-1})u_{n-1}\|\\ &=\|\alpha_n u+(1-\alpha_n)u_n+(1-\alpha_n)u_{n-1}+(1-\alpha_n)u_{n-1}-\alpha_{n-1}u-(1-\alpha_{n-1})u_{n-1}\|\\ &=\|(\alpha_n-\alpha_{n-1})u+(1-\alpha_n)(u_n-u_{n-1})+(\alpha_n-\alpha_{n-1})u_{n-1}\|\\ &\leq|\alpha_n-\alpha_{n-1}|\|u\|+(1-\alpha_n)\|u_n-u_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|u_{n-1}\|.\end{aligned} \tag{3.4}$$

and $u_n=S_{r_n}y_n$ and $u_{n-1}=S_{r_{n-1}}y_{n-1}$, we have

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-y_n\rangle\geq 0,\ \ \forall y\in C, \tag{3.5}$$

and

$$F(u_{n-1}, y) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - y_{n-1}\rangle \geq 0, \ \ \forall y \in C, \tag{3.6}$$

Since $u_n, u_{n-1} \in C$, by (3.5) and (3.6), we have

$$F(u_n, u_{n-1}) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \tag{3.7}$$

and

$$F(u_{n-1}, u_n) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - y_{n-1}\rangle \geq 0. \tag{3.8}$$

From (A2), we have

$$\frac{1}{r_n}\langle u_{n-1} - u_n, u_n - y_n\rangle + \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - y_{n-1}\rangle \geq 0. \tag{3.9}$$

It follows that

$$\left\langle u_n - u_{n-1}, u_{n-1} - y_{n-1} - \frac{r_{n-1}}{r_n}(u_n - y_n)\right\rangle \geq 0. \tag{3.10}$$

Then, we have

$$\left\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - y_{n-1} - \frac{r_{n-1}}{r_n}(u_n - y_n)\right\rangle \geq 0. \tag{3.11}$$

It implies that

$$\begin{aligned}
\|u_{n-1} - u_n\|^2 &\leq \left\langle u_n - u_{n-1}, u_n - y_{n-1} - \frac{r_{n-1}}{r_n}(u_n - y_n)\right\rangle \\
&= \left\langle u_n - u_{n-1}, u_n + y_n - y_{n-1} - \frac{r_{n-1}}{r_n}(u_n - y_n)\right\rangle \\
&= \left\langle u_n - u_{n-1}, y_n - y_{n-1}\left(1 - \frac{r_{n-1}}{r_n}\right)(u_n - y_n)\right\rangle \\
&\leq \|u_n - u_{n-1}\|\left\{\|y_n - y_{n-1}\| + \left|1 - \frac{r_{n-1}}{r_n}\right|\|u_n - y_n\|\right\}.
\end{aligned} \tag{3.12}$$

By (3.12), we have

$$\|u_{n-1}-u_n\| \le \|y_n - y_{n-1}\| + \frac{r_n}{r_{n-1}}\|u_n - y_n\|$$

$$= \|\beta_n Sx_n + (1-\beta_n)x_n - \beta_{n-1}Sx_{n-1} + (1-\beta_{n-1})x_{n-1}\| + \frac{r_n}{r_{n-1}}\|u_n - y_n\|$$

$$= \|\beta_n Sx_n + (1-\beta_n)x_n - (1-\beta_n)x_{n-1} + (1-\beta_n)x_{n-1} - \beta_{n-1}Sx_{n-1} - \beta_n Sx_{n-1} + \beta_n Sx_{n-1}$$

$$+(1-\beta_{n-1})x_{n-1}\| + \frac{r_n}{r_{n-1}}\|u_n - y_n\|$$

$$\le (1-\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \beta_n\|Sx_n - Sx_{n-1}\| + |\beta_{n-1} - \beta_n|\|Sx_{n-1}\|$$

$$+\left|1 - \frac{r_{n-1}}{r_n}\right|\|u_n - y_n\|$$

$$\le \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + |\beta_{n-1} - \beta_n|\|Sx_{n-1}\| + \left|1 - \frac{r_{n-1}}{r_n}\right|\|u_n - y_n\|$$

$$\le \|x_n - x_{n-1}\| + 2M|\beta_{n-1} - \beta_n| + \frac{1}{a}|r_n - r_{n-1}|M, \tag{3.13}$$

where $M = \max_{n\in N}\{\|x_n\|, \|Sx_n\|, \|u_n - y_n\|\}$. Substitute (3.13) into (3.4), we obtain

$$\|x_{n+1} - x_n\| \le |\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\|u_n - u_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|u_{n-1}\|$$

$$\le |\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\left(\|x_n - x_{n-1}\| + 2M|\beta_{n-1} - \beta_n| + \frac{1}{a}|r_n - r_{n-1}|M\right)$$

$$+|\alpha_n - \alpha_{n-1}|\|u_{n-1}\|$$

$$\le |\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\|x_n - x_{n-1}\| + 2M|\beta_{n-1} - \beta_n| + \frac{1}{a}|r_n - r_{n-1}|M$$

$$+|\alpha_n - \alpha_{n-1}|\|u_{n-1}\| \tag{3.14}$$

By (3.14), condition (i), (ii) and Lemma 2.4, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.15}$$

Step 3. We will show that

$$\lim_{n\to\infty}\|Sx_n - x_n\| = 0. \tag{3.16}$$

Since $y_n = \beta_n Sx_n + (1-\beta_n)x_n$, we have

$$\beta_n(Sx_n - x_n) = y_n - x_n. \tag{3.17}$$

Claim that

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0. \tag{3.18}$$

Since $\|x_{n+1} - y_n\| \le \alpha_n\|u - y_n\| + (1-\alpha_n)\|u_n - y_n\|$ (3.19)

and for all $z \in EP(F)\bigcap F(S)$, by firmly nonexpansiveness, we have

$$\|u_n - z\|^2 = \|S_{r_n}y_n - S_{r_n}z\|^2$$

$$\le \langle u_n - z, y_n - z\rangle$$

$$= \frac{1}{2}\left(\|u_n - z\|^2 + \|y_n - z\|^2 - \|u_n - y_n\|^2\right).$$

$$=\frac{1}{2}\left(\|u_n-z\|^2+\|y_n-z\|^2-\|u_n-y_n\|^2\right). \tag{3.20}$$

From (3.20), $\|u_n-z\|^2\le\|y_n-z\|^2-\|u_n-y_n\|^2$. (3.21)

By definition of $\{x_n\}$ and (3.21), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \alpha_n\|u-z\|^2+(1-\alpha_n)\|u_n-z\|^2\\ &\le \alpha_n\|u-z\|^2+(1-\alpha_n)\left(\|y_n-z\|^2-\|u_n-y_n\|^2\right)\\ &= \alpha_n\|u-z\|^2+(1-\alpha_n)\|y_n-z\|^2-(1-\alpha_n)\|u_n-y_n\|^2\\ &\le \alpha_n\|u-z\|^2+(1-\alpha_n)\left((1-\beta_n)\|x_n-z\|^2+\beta_n\|Sx_n-z\|^2\right)\\ &\quad -(1-\alpha_n)\|u_n-y_n\|^2\\ &\le \alpha_n\|u-z\|^2+(1-\alpha_n)\|x_n-z\|^2-(1-\alpha_n)\|u_n-y_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}(1-\alpha_n)\|u_n-y_n\|^2 &\le \alpha_n\|u-z\|^2+\|x_n-z\|^2+\|x_{n+1}-z\|^2\\ &\le \alpha_n\|u-z\|^2+\left(\|x_n-z\|+\|x_{n+1}-z\|\right)\|x_{n+1}-x_n\|.\end{aligned}$$

By condition (i) and (3.15), we have

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.22}$$

By (3.22), condition (i) and (3.19), we obtain (3.18). Again by (3.18) and (3.15), we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.23}$$

By (3.23), (3.17) and $\beta_n\in(c,d]$, we have (3.16).

Step 4. Put $z_0\in\mathbf{F}$, we must show that

$$\lim_{n\to\infty}\langle u-z_0,x_n-z_0\rangle\le 0. \tag{3.24}$$

To show this inequality, choose subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-z_0,x_n-z_0\rangle=\lim_{n\to\infty}\langle u-z_0,x_{n_k}-z_0\rangle \tag{3.25}$$

Without loss of generality, we may assume that $x_{n_k}\rightharpoonup\omega$ as $k\to\infty$ where $\omega\in C$. We first show $\omega\in EP(F)$. By (3.23), we have $y_{n_k}\rightharpoonup\omega$ as $k\to\infty$. Since $u_n=S_{r_n}y_n$, for every $y\in C$, we have

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-y_n\rangle\ge 0, \qquad \forall y\in C,$$

From (A2), we have

$$\frac{1}{r_n}\langle y-u_n,u_n-y_n\rangle\ge F(y,u_n), \qquad \forall y\in C.$$

In particular, we have

$$\left\langle y-u_{n_k}, \frac{u_{n_k}-y_{n_k}}{r_{n_k}} \right\rangle \geq F\left(y, u_{n_k}\right), \qquad \forall y \in C. \tag{3.26}$$

By (3.22) and $y_{n_k} \rightharpoonup \omega$ as $k \to \infty$, we have $u_{n_k} \to \omega$ as $k \to \infty$. Again by (3.22), (3.26) and $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$, we have

$$0 \geq F(y,\omega), \qquad \forall y \in C \tag{3.27}$$

Let $t \in (0,1]$ and $y \in C$, put $y_t = ty + (1-t)\omega$, we have $y_t \in C$. By (3.27)

$$0 \geq F(y_t,\omega), \qquad \forall y \in C \tag{3.28}$$

By (A1), (A4) and (3.28), we have

$$\begin{aligned} 0 = F\left(y_t, y_t\right) &\leq tF\left(y_t, y\right) + (1-t)F\left(y_t,\omega\right) \\ &\leq tF\left(y_t, y\right). \end{aligned}$$

It follows that

$$0 \leq F\left(y_t, y\right). \tag{3.29}$$

From (A3) and (3.29), we have

$$0 \leq F\left(\omega, y\right),$$

Then $\omega \in EP(F)$. By $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$, (3.16) and Lemma 2.3, we have $\omega \in F(T)$. Hence $\omega \in \mathbf{F}$.

By (3.25), we have

$$\begin{aligned} \limsup_{n\to\infty} \left\langle u-z_0, x_n - z_0 \right\rangle &= \lim_{n\to\infty} \left\langle u-z_0, x_{n_k} - z_0 \right\rangle \\ &= \left\langle u-z_0, \omega - z_0 \right\rangle \leq 0 \end{aligned}$$

Step 5. Finally, we show that $x_n \to \omega$ as $n \to \infty$, where $z_0 = P_{\mathbf{F}}u$. By nonexpansiveness of S and S_{r_n}, we have

$$\begin{aligned} \left\|x_{n+1} - z_0\right\|^2 &= \left\|\alpha_n (u-z_0) + \left(1-\alpha_n\right)(u_n - z_0)\right\|^2 \\ &\leq \left(1-\alpha_n\right)^2 \left\|u_n - z_0\right\|^2 + 2\alpha_n \left\langle u-z_0, x_{n+1}-z_0\right\rangle \\ &\leq \left(1-\alpha_n\right)\left\|s_{r_n} y_n - z_0\right\|^2 + 2\alpha_n \left\langle u-z_0, x_{n+1}-z_0\right\rangle \\ &\leq \left(1-\alpha_n\right)\left\|y_n - z_0\right\|^2 + 2\alpha_n \left\langle u-z_0, x_{n+1}-z_0\right\rangle \\ &= \left(1-\alpha_n\right)\left\|\beta_n (Sx_n - z_0) + \left(1-\beta_n\right)\left(x_n - z_0\right)\right\|^2 + 2\alpha_n \left\langle u-z_0, x_{n+1}-z_0\right\rangle \\ &\leq \left(1-\alpha_n\right)\left(\beta_n \left\|sx_n - z_0\right\|^2 + \left(1-\beta_n\right)\left\|x_n - z_0\right\|^2\right) + 2\alpha_n \left\langle u-z_0, x_{n+1}-z_0\right\rangle \\ &\leq \left(1-\alpha_n\right)\left\|x_n - z_0\right\|^2 + 2\alpha_n \left\langle u-z_0, x_{n+1}-z_0\right\rangle \end{aligned}$$

From step 4, condition (i) and lemma 2.1, we can conclude that $\{x_n\}$ converges strongly to $z_0 = P_{\mathbf{F}}u$. This completes the proof.

Conclusion

We define a new iterative scheme to generate the sequence $\{x_n\}$ in theorem 3.1 and prove that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathbf{F}}u.$ This paper is to solve the equilibrium problem and fixed point problem of strictly pseudocontractive mapping.

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