Strong Convergence of a New Iteration for Fixed Point Problems of a Finite Family of Nonexpansive Mappings and Equilibrium Problems

การลู่เข้าของกระบวนการทำซ้ำแบบใหม่สำหรับปัญหาจุดตรึงของวงศ์จำกัดของการส่งแบบไม่ขยายและปัญหาเชิงดุลยภาพ

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ABSTRACT

In this paper, we establish a new iterative method for fixed point problem of a finite family of nonexpansive mappings and the equilibrium problem. Then, we prove a strong convergence theorem for finding the common element of these problems.

บทคัดย่อ

งานวิจัยนี้ ผู้วิจัยได้สร้างกระบวนการทำซ้ำแบบใหม่สำหรับปัญหาจุดตรึงของวงศ์จำกัดของการส่งแบบไม่ขยาย และปัญหาดุลยภาพ นอกจากนั้นยังได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มสำหรับการหาสมาชิกร่วมของปัญหาที่ได้กล่าวมาข้างต้น

Keywords: Nonexpansive mapping, S – mapping, Equilibrium problem, Fixed point

คำสำคัญ: การส่งแบบไม่ขยาย การส่งแบบ S ปัญหาดุลยภาพ ปัญหาจุดตรึง

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Introduction

Throughout this paper, we always assume that H is a real Hilbert space. Let C be a nonempty closed convex subset of H. Let $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively. Let P_C be the metric projection of H onto C. i.e. for $x \in H, P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\| .$$

P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 .$$

The set of fixed point of a mapping $S : C \to C$ is denoted by $F(S)$, that is, $F(S) = \{x \in C : Sx = x\}$. Goebel and Kirk [5] showed that $F(S)$ is always closed convex, and also nonempty provided S has a bounded trajectory. Recall that S is said to be nonexpansive mapping if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in H$.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of the of (1.1) is denoted by $EP(F)$. In 2005, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

In 2007, Takahashi and Takahashi [8] introduced viscosity approximation method in framework of a real Hilbert space H. They defined the iterative sequence $\{x_n\}$ and $\{u_n\}$ as follows:

$$\begin{cases} x_1 \in H, \\ F(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T u_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $f : H \to H$ is a contraction mapping with constant $\alpha \in (0,1)$ and $\{\alpha_n\} \subset [0,1], \{r_n\} \subset (0, \infty)$. They proved under some suitable conditions on the sequences $\{\alpha_n\}, \{r_n\}$ and the bifunction F that $\{x_n\}, \{u_n\}$ strongly converges to $z \in F(T) \bigcap EP(F)$, where $z = P_{F(T) \bigcap EP(F)} f(z)$.

Inspired by Takahashi and Takahashi [8], we introduced an iterative method and proved a strong convergence theorem for finding the solution of fixed point problems of a finite family of nonexpansive mappings and modified equilibrium problem under some suitable conditions.

Objective of the study

The purpose of this research was to introduce a new iterative scheme for fixed point and equilibrium problems and proof a new strong convergence theorems of a finite family of nonexpansive mappings in Hilbert space.

Preliminaries

In this section, we give some useful lemmas and definitions that will be needed for our main result.

Lemma 2.1 [9] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \le (1-\alpha_n)s_n + \beta_n, \quad \forall n \ge 0$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions

(1) $\{\alpha_n\} \subset (0,1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \le 0$ or $\sum_{n=1}^{\infty} |\beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.2 [3] Let E be a uniformly convex Banach space, C be nonempty closed convex subset of E and $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

$(A1)$ $F(x,x) = 0 \quad \forall x \in C$;

$(A2)$ F is monotone, i.e, $F(x,y) + F(y,x) \le 0, \quad \forall x, y \in C$;

$(A3)$ $\forall x, y, z \in C, \quad \lim_{t\to 0^+} F(tz + (1-t)x, y) \le F(x,y)$;

$(A3)$ $\forall x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.3 [1] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1) - (A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z,y) + \frac{1}{r}\langle y - z, z - x \rangle, \tag{2.1}$$

for all $x \in C$.

Lemma 2.4 [4] Let C be a nonempty closed convex subset of H. Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $J_r : H \to C$ as follows:

$$J_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in C \right\}, \tag{2.2}$$

for all $z \in H$. Then, the following hold:

(1) J_r is single – valued;

(2) J_r is firmly nonexpansive i.e. $\|J_r(x) - J_r(y)\|^2 \leq \langle J_r(x) - J_r(y), x - y \rangle \quad \forall x, y \in H$;

(3) $F(J_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

Lemma 2.5 [7] Let C, H, F and $J_r(x)$ be as in Lemma 2.4. Then the following holds:

$$\|J_s x - J_t x\|^2 \leq \frac{s - t}{s}\langle J_s x - J_t x, J_s x - x \rangle$$

for all $s, t > 0$ and $x \in H$.

Definition 2.1 [6] Let C be a nonempty closed convex subset of a real Banach space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpanxive mappings of C into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = \left(\alpha_1^j, \alpha_2^j, \alpha_3^j\right) \in I \times I \times I$ where $I = [0,1]$ and $, \alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. We define the mapping $S : C \to C$ as follow:

$$\begin{aligned}
U_0 &= I \\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I \\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I \\
U_3 &= \alpha_1^3 T_1 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I \\
&\ \ . \\
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&\ \ . \\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I \\
S &= U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called the S - mapping generated by $T_1, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

Lemma 2.6 [6] Let C be a nonempty closed convex subset of a strictly convex. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpanxive mappings of C into itself with $F=\bigcap_{i=1}^{N}F(T_i)\neq\varnothing$ and let $\alpha_j=\left(\alpha_1^j,\alpha_2^j,\alpha_3^j\right)\in I\times I\times I$, $j=1,2,3,...,N$, where $I=[0,1]$,$\alpha_1^j+\alpha_2^j+\alpha_3^j=1$, $\alpha_1^j\in(0,1)$ for all $j=1,2,...,N-1$, $\alpha_1^N\in(0,1]$, $\alpha_1^N,\alpha_1^N\in[0,1)$ for all $j=1,2,...,N$. Let S be the mapping generated by $T_1,...,T_N$ and $\alpha_1,\alpha_2,...,\alpha_N$. Then $F(S)=\bigcap_{i=1}^{N}F(T_i)$.

Lemma 2.7 [2] Let C be a nonempty closed convex subset of a Banach space E. Let F be a bifunction from $C\times C\to\mathbb{R}$ satisfying (A1)–(A4). Suppose that $p\in C.$ Then $p\in EP(F).$ if and only if $F(y,p)\leq 0$ for all $y\in C.$

Main result

Theorem 3.1 Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunctions from $C\times C$ into $\mathbb{R}$ satisfying $(A_1)-(A_4)$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpanxive mappings of C into itself with $\mathfrak{F}=\bigcap_{i=1}^{N}F(T_i)\bigcap EP(F)\neq\varnothing$ and let $\alpha_j=\left(\alpha_1^j,\alpha_2^j,\alpha_3^j\right)\in I\times I\times I$, $j=1,2,3,...,N$, where $I=[0,1]$,$\alpha_1^j+\alpha_2^j+\alpha_3^j=1$, $\alpha_1^j\in(0,1)$ for all $j=1,2,...,N-1$, $\alpha_1^N\in(0,1]$ $\alpha_1^N,\alpha_1^N\in[0,1)$ for all $j=1,2,...,N$. Let S be the mapping generated by $T_1,...,T_N$ and $\alpha_1,\alpha_2,...,\alpha_N$. Let $\{x_n\}$ be a sequence generated by $x_1,u\in C$:

$$\begin{cases} F\left(u_n,y\right)+\dfrac{1}{r_n}\left\langle y-u_n,u_n-\left(\beta_n Sx_n+\left(1-\beta_n\right)x_n\right)\right\rangle\geq 0, & \forall y\in C,\\ x_{n+1}=\alpha_n u+(1-\alpha_n)u_n, & \forall n\geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\},\{\beta_n\}\in(0,1)$ such that $r_n\in(a,b)$ and $\beta_n\in(c,d]\subset(0,1]$.

Assume that

(i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$,

(ii) $\sum_{n=1}^{\infty}\left|r_{n+1}-r_n\right|,\sum_{n=1}^{\infty}\left|\alpha_{n+1}-\alpha_n\right|,\sum_{n=1}^{\infty}\left|\beta_{n+1}-\beta_n\right|<\infty.$

Then the sequence $\{x_n\}$ converges strongly to $z=P_{\mathfrak{F}}u.$

Proof. We divide our proof into 5 steps.

Step 1. We show that $\{x_n\}$ is bounded.

Let $z \in \mathfrak{F}$, since $F\left(u_n, y\right)+\frac{1}{r_n}\left\langle y-u_n, u_n-\left(\beta_n S x_n+\left(1-\beta_n\right) x_n\right)\right\rangle \geq 0, \quad \forall y \in C,$ (3.2)

by Lemma 2.4, it implies that $u_n = J_{r_n}\left(\beta_n T x_n+\left(1-\beta_n\right) x_n\right)$ and $z \in F(J_{r_n})$.

Then, we have

$$
\begin{aligned}
\left\|x_{n+1}-z\right\| &\leq \alpha_n\|u-z\|+(1-\alpha_n)\left\|u_n-z\right\| \\
&= \alpha_n\|u-z\|+(1-\alpha_n)\left\|J_{r_n}\left(\beta_n S x_n+\left(1-\beta_n\right) x_n\right)-z\right\| \\
&\leq \alpha_n\|u-z\|+(1-\alpha_n)\left\|\left(\beta_n S x_n+\left(1-\beta_n\right) x_n\right)-z\right\| \\
&\leq \alpha_n\|u-z\|+(1-\alpha_n)\left(\beta_n\left\|S x_n-z\right\|+\left(1-\beta_n\right)\left\|x_n-z\right\|\right) \\
&\leq \alpha_n\|u-z\|+(1-\alpha_n)\left(\beta_n\left\|x_n-z\right\|+\left(1-\beta_n\right)\left\|x_n-z\right\|\right) \\
&= \alpha_n\|u-z\|+(1-\alpha_n)\left\|x_n-z\right\| \\
&\leq \max\left\{\|u-z\|,\left\|x_n-z\right\|\right\}.
\end{aligned} \tag{3.3}
$$

By induction, we can conclude that $\{x_n\}$ is bounded and so are $\{Sx_n\}$ and $\{u_n\}$.

Step 2. We show that $\lim_{n\to\infty}\left\|x_{n+1}-x_n\right\| = 0.$

Put $y_n = \beta_n S x_n + (1-\beta_n)x_n$, for all $n \in \mathbb{N}$.

We see that

$$
\begin{aligned}
\left\|x_{n+1}-x_n\right\| &= \left\|\alpha_n u+(1-\alpha_n)u_n-\alpha_{n-1}u-(1-\alpha_{n-1})u_{n-1}\right\| \\
&= \left\|\alpha_n u+(1-\alpha_n)u_n-(1-\alpha_n)u_{n-1}+(1-\alpha_n)u_{n-1}-\alpha_{n-1}u-(1-\alpha_{n-1})u_{n-1}\right\| \\
&= \left\|(\alpha_n-\alpha_{n-1})u+(1-\alpha_n)(u_n-u_{n-1})+(\alpha_n-\alpha_{n-1})u_{n-1}\right\| \\
&\leq \left|\alpha_n-\alpha_{n-1}\right|\|u\|+(1-\alpha_n)\left\|u_n-u_{n-1}\right\|+\left|\alpha_n-\alpha_{n-1}\right|\left\|u_{n-1}\right\|.
\end{aligned} \tag{3.4}
$$

Since $u_n = J_{r_n} y_n$, $u_{n-1} = J_{r_{n-1}} y_{n-1}$ and from Lemma 2.5, we obtain

$$\begin{aligned}\|u_n - u_{n-1}\|^2 &\le \left(1-\frac{r_{n-1}}{r_n}\right)\langle u_n - u_{n-1}, u_n - y_n\rangle \\ &\le \left|1-\frac{r_{n-1}}{r_n}\right| \|u_n - u_{n-1}\|\|u_n - y_n\|.\end{aligned}$$

It implies that $\|u_n - u_{n-1}\| \le \left|1-\dfrac{r_{n-1}}{r_n}\right| \|u_n - y_n\|.$ (3.12)

By (3.12), we have

$$\begin{aligned}\|u_n - u_{n-1}\| &\le \|y_n - y_{n-1}\| + \left|1-\frac{r_{n-1}}{r_n}\right|\|u_n - y_n\| \\ &= \|\beta_n Sx_n + (1-\beta_n)x_n - \beta_{n-1}Sx_{n-1} + (1-\beta_{n-1})x_{n-1}\| + \left|1-\frac{r_{n-1}}{r_n}\right|\|u_n - y_n\| \\ &= \|\beta_n Sx_n + (1-\beta_n)x_n - (1-\beta_n)x_{n-1} + (1-\beta_n)x_{n-1} - \beta_{n-1}Sx_{n-1} - \beta_n Sx_{n-1} + \beta_n Sx_{n-1} \\ &\quad +(1-\beta_{n-1})x_{n-1}\| + \left|1-\frac{r_{n-1}}{r_n}\right|\|u_n - y_n\| \\ &\le (1-\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \beta_n\|Sx_n - Sx_{n-1}\| + |\beta_{n-1} - \beta_n|\|Sx_{n-1}\| \\ &\quad + \left|1-\frac{r_{n-1}}{r_n}\right|\|u_n - y_n\| \\ &\le \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + |\beta_{n-1} - \beta_n|\|Sx_{n-1}\| + \left|1-\frac{r_{n-1}}{r_n}\right|\|u_n - y_n\| \\ &\le \|x_n - x_{n-1}\| + 2M\,|\beta_{n-1} - \beta_n| + \frac{1}{a}|r_n - r_{n-1}|\,M, \qquad (3.13)\end{aligned}$$

where $M = \max_{n\in\mathbb{N}}\{\|x_n\|, \|Sx_n\|, \|u_n - y_n\|\}$.

Substitute (3.13) into (3.4)

$$\begin{aligned}\|x_{n+1} - x_n\| &\le |\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\|u_n - u_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|u_{n-1}\| \\ &\le |\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\left(\|x_n - x_{n-1}\| + 2M\,|\beta_{n-1} - \beta_n| + \frac{1}{a}|r_n - r_{n-1}|\,M\right) \\ &\quad + |\alpha_n - \alpha_{n-1}|\|u_{n-1}\| \\ &\le |\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\|x_n - x_{n-1}\| + 2M\,|\beta_{n-1} - \beta_n| + \frac{1}{a}|r_n - r_{n-1}|\,M \\ &\quad + |\alpha_n - \alpha_{n-1}|\|u_{n-1}\|. \qquad (3.14)\end{aligned}$$

This together with the conditions $(i),(ii)$ and Lemma 2.1, gives

$$\lim_{n\to\infty}\left\|x_{n+1}-x_n\right\|=0. \tag{3.15}$$

Step 3. We show that $\lim_{n\to\infty}\left\|Sx_n-x_n\right\|=0.$ (3.16)

Since $y_n=\beta_n Sx_n+(1-\beta_n)x_n$, we have $\beta_n\left(Sx_n-x_n\right)=y_n-x_n.$ (3.17)

Claim that $\lim_{n\to\infty}\left\|x_{n+1}-y_n\right\|=0.$ (3.18)

Since $\left\|x_{n+1}-y_n\right\|\le\alpha_n\left\|u-y_n\right\|+\left(1-\alpha_n\right)\left\|u_n-y_n\right\|.$ (3.19)

For $z\in EP(F)\cap F(S)$, by firmly nonexpansiveness, we have

$$\begin{aligned}\left\|u_n-z\right\|^2&=\left\|J_{r_n}y_n-J_{r_n}z\right\|^2\\&\le\left\langle u_n-z,y_n-z\right\rangle\\&=\frac{1}{2}\left(\left\|u_n-z\right\|^2+\left\|y_n-z\right\|^2-\left\|u_n-y_n\right\|^2\right).\end{aligned} \tag{3.20}$$

From (3.20), we have $$\left\|u_n-z\right\|^2\le\left\|y_n-z\right\|^2-\left\|u_n-y_n\right\|^2. \tag{3.21}$$

By definition of $\left\{x_n\right\}$ and (3.21), we have

$$\begin{aligned}\left\|x_{n+1}-z\right\|^2&\le\alpha_n\left\|u-z\right\|^2+\left(1-\alpha_n\right)\left\|u_n-z\right\|^2\\&\le\alpha_n\left\|u-z\right\|^2+\left(1-\alpha_n\right)\left(\left\|y_n-z\right\|^2-\left\|u_n-y_n\right\|^2\right)\\&=\alpha_n\left\|u-z\right\|^2+\left(1-\alpha_n\right)\left\|y_n-z\right\|^2-\left(1-\alpha_n\right)\left\|u_n-y_n\right\|^2\\&\le\alpha_n\left\|u-z\right\|^2+\left(1-\alpha_n\right)\left((1-\beta_n)\left\|x_n-z\right\|^2+\beta_n\left\|Sx_n-z\right\|^2\right)-\left(1-\alpha_n\right)\left\|u_n-y_n\right\|^2\\&\le\alpha_n\left\|u-z\right\|^2+\left(1-\alpha_n\right)\left\|x_n-z\right\|^2-\left(1-\alpha_n\right)\left\|u_n-y_n\right\|^2,\end{aligned}$$

it follows that

$$\begin{aligned}\left(1-\alpha_n\right)\left\|u_n-y_n\right\|^2&\le\alpha_n\left\|u-z\right\|^2+\left\|x_n-z\right\|^2+\left\|x_{n+1}-z\right\|^2\\&\le\alpha_n\left\|u-z\right\|^2+\left(\left\|x_n-z\right\|+\left\|x_{n+1}-z\right\|\right)\left\|x_{n+1}-x_n\right\|.\end{aligned}$$

By condition (i) and (3.15), we have $\lim_{n\to\infty}\|u_n - y_n\| = 0.$ (3.22)

By (3.22), condition (i) and (3.19), we obtain (3.18). Again by (3.18) and (3.15), we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.23}$$

By (3.23), (3.17) and $\beta_n \in (c,d]$, we have $\lim_{n\to\infty}\|Sx_n - x_n\| = 0.$

Step 4. We show that

$$\lim_{n\to\infty}\langle u - z_0, x_n - z_0\rangle \le 0 \tag{3.24}$$

where $z_0 \in \mathfrak{F}$. Indeed, we pick a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty}\sup\langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty}\langle u - z_0, x_{n_k} - z_0\rangle. \tag{3.25}$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$. Since C is closed and convex, C is weakly closed. So, we obtain $\omega \in C$.

First, we show that $\omega \in EP(F)$. By (3.23), we have $y_{n_k} \rightharpoonup \omega$ as $k \to \infty$. Since as $u_n = J_{r_n} y_n$, for all $y \in C$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \ge 0, \qquad \forall y \in C.$$

From (A2), we have

$$\frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \ge F(y, u_n), \qquad \forall y \in C.$$

In particular, we have

$$\left\langle y - u_{n_k}, \frac{u_{n_k} - y_{n_k}}{r_{n_k}}\right\rangle \ge F\left(y, u_{n_k}\right), \qquad \forall y \in C. \tag{3.26}$$

By (3.22) and $y_{n_k} \rightharpoonup \omega$ as $k \to \infty$, we have $u_{n_k} \to \omega$ as $k \to \infty$. Again by (3.22), (3.26) and $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$, we have

$$F(y, \omega) \le 0, \qquad \forall y \in C. \tag{3.27}$$

From Lemma 2.7 and (3.27), we have $\omega \in EP(F)$.

By $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$, (3.16) and Lemma 2.3, we have $\omega \in F(S)$. Thus $\omega \in \mathfrak{F}$.

By (3.25), we obtain

$$\begin{aligned}\lim_{n\to\infty} \sup \langle u - z_0, x_n - z_0 \rangle &= \lim_{n\to\infty} \langle u - z_0, x_{n_k} - z_0 \rangle \\ &= \langle u - z_0, \omega - z_0 \rangle \\ &\le 0.\end{aligned}$$

Step 5. Finally, we show that $x_n \to z_0$ as $n \to \infty$, where $z_0 = P_{\mathfrak{F}} u$.

By nonexpansiveness of S and J_{r_n}, we have

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &= \|\alpha_n (u - z_0) + (1-\alpha_n)(u_n - z_0)\|^2 \\ &\le (1-\alpha_n)^2 \|u_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &\le (1-\alpha_n) \|J_{r_n} y_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &\le (1-\alpha_n) \|y_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &= (1-\alpha_n) \|\beta_n (Sx_n - z_0) + (1-\beta_n)(x_n - z_0)\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &\le (1-\alpha_n) \left(\beta_n \|Sx_n - z_0\|^2 + (1-\beta_n) \|x_n - z_0\|^2 \right) + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle \\ &\le (1-\alpha_n) \|x_n - z_0\|^2 + 2\alpha_n \langle u - z_0, x_{n+1} - z_0 \rangle.\end{aligned}$$

From step 4, condition (i) and Lemma 2.1, we can conclude $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}} u$.

This completes the proof.

Conclusion

Theorem 3.1 tells us that the sequence $\{x_n\}$, generated by the iterative (3.1), converges strongly to $z = P_{\mathfrak{F}} u$. Moreover, this point is a common solution of fixed point problem of a finite family of nonexpansive mappings and the equilibrium problem. Therefore, we can apply theorem 3.1 to solve the problem that accordance with the conditions of this theorem.

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References

Blum E, Oettli W. From optimization and variational inequalities to equilibrium problems, Math. Stud. 63; 1994, 123-145.

Boonchari D, Saejung S. Approximation of common fixed point of a countable family of relatively nonexpansive mappings. Fixed Point Theory Appl. 2010, Art. ID 407651, 26 pp.

Browder F.E. Nonlinear operators and nonlinear equations of evolution in Banach spaces, Proc. Sympos. Pure. Math. 18; 1976, 78-81.

Combettes P.L, Hirstoaga A. Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6; 2005, 117-136.

Goebel K, W.A. Kirk. Topics in Metric Fixed Point Theory, Cambridge Stud. Adv. Math., Vol. 28, Cambridge University Press, Cambridge; 1990.

Kangtunyakarn A, Suantai S. Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings, Nonlinear Analysis: Hybrid Systems 3; 2009, 296-309.

Takahashi S, Takahashi W. Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space. Nonlinear Anal. 69; 2008, no.3, 1025-1033.

Takahashi S, Takahashi W. Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331; 2007, 506-515.

Xu H.K. Another control condition in an iterative method for nonexpansive mappings, Bull. Austral. Math. Soc. 65; 2002, 109-113.